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2^m fractional factorial designs of resolution V with high A-efficiency, $7 \leqslant m \leqslant 10$ [1]

Nam-Ky Nguyen*, Alan J. Miller

CSIRO Division of Mathematics & Statistics, Private Bag 10, Clayton South MDC, Victoria 3169, Australia

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Abstract

We present 111 2^m fractional factorial designs of resolution V for $7 \leqslant m \leqslant 10$. These designs are the best known to the authors with respect to the A-optimality criterion (as of October 1995).

AMS subject classification: 62K05, 62K15

Keywords: A-optimality; Computer-aided designs; Fractional factorial designs; Resolution V

1. Introduction

Srivastava and/or Chopra in a series of papers (Srivastava and Chopra 1971; Chopra and Srivastava 1973a, b, 1974, 1975; Chopra, 1975a, b, 1977a, b, 1979, 1983) have presented A-optimal balanced 2^m fractional factorial (2^m-BFF) designs of resolution V for $m \leqslant 10$ for practical values of n (the total number of runs or level-combinations) in the class of *balanced* designs. The balance property of these designs leads to ease in the analysis and interpretation of the results. Since 2^m-BFF designs only form a subclass of *all* 2^m fractional factorial (2^m-FF) designs balanced and not balanced, Professor

* Corresponding author. Tel.: +61 395422258; Fax: +61 395436613; e-mail: namky@forprod.csiro.au.

[1] Editor's note: Although the authors of this paper do not guarantee the optimality of their designs, they believe that for a given value of (m, n), their design has a very high value of the absolute efficiency E, among all *currently known* designs. If at any time, for certain value(s) of (m, n), any one has obtained a design with an appreciably higher value of E (in particular values of (m, n) for which the authors fail to improve the designs of Srivastava and/or Chopra), then he/she is invited to send a short note to that effect, for publication in the Discussion Forum of JSPI. J.N. Srivastava, Editor-in- Chief, JSPI.

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Srivastava suggested research in optimal designs without the balance constraint and this is the theme of this note.

2. Method

The model for the 2^m-FF designs of resolution V is the full rank linear model $y = X\beta + e$. The ith row of the design matrix X is a p-dimensional row vector $x_i' = (1, x_{i1}, x_{i2}, \ldots, x_{im}, x_{i1}x_{i2}, \ldots, x_{im-1}x_{im})$ where $p = 1 + m + \binom{m}{2}$ and $x_{ij} = \pm 1$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. The problem is for a given (m, n) pair, we have to choose n vectors out of 2^m candidate vectors such that trace V where $V = (X'X)^{-1}$ is minimized. Our approach for this problem is to construct at least 100 designs for each (m, n) pair using a Fedorov exchange algorithm reported in Nguyen and Miller (1992) or Miller and Nguyen (1994) with the D-optimality criterion as the objective function and to pick the best one among these designs with respect to the A-optimality criterion. The rationale of this approach is that the three A-, D- and E-optimality criteria usually imply each other (Srivastava and Anderson, 1974) and it is much easier to construct D-optimal designs using the simple updating formulae in the D-optimality criterion (Nguyen and Miller, 1992).

3. An example

To give readers the feel of our design, we present our 2^{10}-FF design in 76 runs where the + and - represent +1 and -1:

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This design has A-efficiency E=86.3 while the corresponding BFF design has E=28.2 (Chopra, et al. 1986). The variance of the mean of this design is 0.013731. The variances of the main effects range from 0.014736 to 0.015369. The variances of 2-factor interactions range from 0.014736 to 0.019691. For the corresponding BFF design, the variances of the mean, each main effect and each 2-factor interaction are 0.063903,

Table 1
2^7-FF designs

n	tr V	E
29	1.33403	75.0
30	1.27847	75.6
31	1.22063	76.6
32	1.16950	77.5
33	1.10552	79.5
34	1.00771	84.6
35	0.98003	84.6
36	0.91720	87.8
37	0.89530	87.5
38	0.87220	87.5
39	0.83838	88.7
40	0.78993	91.8
41	0.77566	91.2
42	0.76071	90.8
43	0.74506	90.5
44	0.72577	90.8
45	0.71521	90.1
46	0.69635	90.5
47	0.67927	90.8
48	0.65813	91.8
49	0.62932	94.0
50	0.61003	95.1
51	0.59074	96.3
52	0.58290	95.7
53	0.57841	94.6
54	0.56625	94.8
55	0.55483	95.0
56	0.54340	95.3
57	0.53487	95.1
58	0.52139	95.9
59	0.51190	96.0
60	0.50223	96.2
61	0.49094	96.8
62	0.47906	97.6
63	0.46607	98.8
64	0.45313	100.0
65	0.44825	99.5
66	0.44338	99.1
67	0.43854	98.7
68	0.43369	98.3

0.030147 and 0.049988, respectively. Clearly, the variances of our design particularly those of the main effects are within a very small range and *uniformly* smaller than the variances of the corresponding BFF design. Our design, like most of our designs presented in Tables 1–4, could be gainfully employed in situations where the experiment does not require a design which provides different estimates with the same accuracy and precision.

The program used to generate the design points of the designs listed in Tables 1–4 is available from the authors.

Table 2
2^8-FF designs

n	tr V	E
37	1.62968	61.4
38	1.52706	63.8
39	1.40943	67.3
40	1.30268	71.0
41	1.23794	72.9
42	1.17263	75.1
43	1.15422	74.6
44	1.10282	76.3
45	1.06627	77.1
46	1.02936	78.1
47	0.97785	80.5
48	0.95061	81.1
49	0.90838	83.1
50	0.88642	83.5
51	0.85434	84.9
52	0.82273	86.5
53	0.79694	87.6
54	0.77353	88.6
55	0.74979	89.7
56	0.72780	90.8
57	0.71237	91.1
58	0.69509	91.8
59	0.67671	92.7
60	0.65765	93.8
61	0.64096	94.6
62	0.62046	96.2
63	0.59954	98.0
64	0.57813	100.0
65	0.57240	99.5

Table 3
2^9-FF designs

n	tr V	E
46	1.84786	54.1
47	1.65502	59.1
48	1.60334	59.8
49	1.52109	61.7
50	1.41917	64.8
51	1.35161	66.7
52	1.26460	69.9
53	1.24759	69.6
54	1.19824	71.1
55	1.14161	73.3
56	1.10392	74.4
57	1.06296	75.9
58	1.02959	77.0
59	0.99578	78.3
60	0.96196	79.7
61	0.93375	80.8
62	0.90648	81.9
63	0.88093	82.9
64	0.85752	83.8
65	0.83541	84.7
66	0.81587	85.4

Table 4
2^{10}-FF designs

n	tr V	E
56	2.00015	50.0
57	1.88900	52.0
58	1.69936	56.8
59	1.65498	57.4
60	1.56906	59.5
61	1.46465	62.7
62	1.42944	63.2
63	1.35698	65.5
64	1.31586	66.5
65	1.25313	68.8
66	1.20820	70.2
67	1.16489	71.8
68	1.11964	73.6
69	1.08609	74.7
70	1.05110	76.1
71	1.01380	77.8
72	0.97776	79.6
73	0.95766	80.1
74	0.92676	81.7
75	0.86956	85.9
76	0.85427	86.3

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