Incomplete block designs

Walter T. Federer & Nam-Ky Nguyen

Volume 2, pp 1039–1042

in

Encyclopedia of Environmetrics
(ISBN 0471 899976)

Edited by

Abdel H. El-Shaarawi and Walter W. Piegorsch

Incomplete block designs

The subject of statistics deals with variability and how to deal with it. In the planning and conduct of an environmental or ecological investigation, the items used to control variability are (a) refinement of experimental technique, (b) selection of homogeneous material and/or environments, (c) grouping (blocking, stratifying) material into homogeneous subgroups (blocks, strata), and (d) measurement of related variables and use of covariance. Item (c) is an application of the Fisherian principle of local control. The second Fisherian principle of replication is used to reduce further the variability of estimates. The third Fisherian principle of randomization provides for unbiased estimates of effects and their variances.

There are many ways of blocking (arranging) the experimental units (EUs) in a comparative experiment with v treatments. If the sample of EUs is from a homogeneous population, then no blocking is required and a completely randomized experiment design (ED) of the v treatments randomly allotted to the rv EUs is used. The replicate number (sample size) for each treatment is r unless unequal replication is desired. If homogeneous blocks of size v are available to accommodate all v treatments, a randomized complete block ED (all v treatments in each block, not necessarily an equal number of times) is used. In many situations, the block size, k, is less than v and an incomplete block ED (not all treatments in each of the blocks) is used. When $v = ks$, a complete block of size v can be divided into s incomplete blocks of size k each. For r complete blocks of this type the ED is denoted as a resolvable ED. Yates [2] first described incomplete block experiment designs (IBEDs). There are many types, and a large literature exists for IBEDs. Indeed, several methods of constructing IBEDs are available.

A simple method of constructing 0 and 1 concurrence (the number of times a pair of treatments occurs together) in incomplete block designs is the diagonalization method presented in [6, 15]. For $v = ks$, $k < s$, the following steps are used to construct resolvable IBEDs (all v treatments occur in a complete block of size v, i.e. s incomplete blocks form a complete block):

1. Array the v treatments in s rows and k columns, with the rows determining the s incomplete blocks for replicate 1 (complete block).
2. Starting in the upper left corner of replicate 1, use the main right diagonal for the first incomplete block of replicate 2. The first column of replicate 2 is the same as the first column of replicate 1. Serially permute the numbers in the remaining $k - 1$ columns.
3. The first incomplete block of replicate 3 is the main right diagonal of replicate 2 and the replicate is completed as in step 2.
4. Continue the process until the desired r replicates have been obtained.

To illustrate, suppose $v = 24$. The block size k could be 2, 3, or 4. Use $k = 3$ and $s = 8$. The rows are the incomplete blocks. The unrandomized replicate plans (complete blocks) are shown in Table 1. The concurrence of pairs of treatments in replicates 1 to 4 is 0 or 1. Some pairs begin to appear together twice in replicates 5 and 6. A better design in the sense of balance and retaining the 0-1 concurrence pattern would have been to use 1, 13, and 18 as the first block of replicate 5. Although the intrablock efficiency measure of efficiency would be better for a 0-1 concurrence than for a 0-1-2 concurrence, the efficiency measure recovering interblock information (mixed model) is relatively unaffected [10]. If incomplete block sizes of k and $k - 1$ are allowed, then the above method works for any v and not just for $v = ks$. For example, in the above, if there had been only 23 treatments, then we would use the above design and eliminate the number 24 to obtain the plan.

The block size k need not be less than v. There are situations where homogeneous EUs within blocks can be obtained for $k > v$ [5]. To illustrate, consider the generalized incomplete block designs of Shafiq and Federer [23] where the concurrences may be m, $m + 1$, $m + 2$, etc. To illustrate, let $v = 3$ and $k = 8$ (this was the case for a nutrition experiment using eight male rats from a single litter with three treatments A, B, and C). A balanced (single concurrence number for every pair) block design for $v = 3$, $k = 8$, $r = 8$, $\lambda = 21$ (where each pair of letters occurs together 21 times) in three blocks is

AAABBBCC BBBCCCAA CCCAAABB

Other designs of this type may be obtained by adding rows to a Latin square and using the columns as blocks.

Although many other methods of constructing designs are available (see, for example, [21] and [22])

Table 1

Replicate 1			Replicate 2			Replicate 3			Replicate 4			Replicate 5			Replicate 6		
1	9	17	1	10	19	1	11	21	1	12	23	1	13	17	1	14	19
2	10	18	2	11	20	2	12	22	2	13	24	2	14	18	2	15	20
3	11	19	3	12	21	3	13	23	3	14	17	3	15	19	3	16	21
4	12	20	4	13	22	4	14	24	4	15	18	4	16	20	4	9	22
5	13	21	5	14	23	5	15	17	5	16	19	5	9	21	5	10	23
6	14	22	6	15	24	6	16	18	6	9	20	6	10	22	6	11	24
7	15	23	7	16	17	7	9	19	7	10	21	7	11	23	7	12	17
8	16	24	8	9	18	8	10	20	8	11	22	8	12	24	8	13	18

there are computer software packages and toolkits which will construct optimal or near-optimal block designs and row-column designs (e.g. [12, 16–20]). The incomplete block modules in a toolkit use an iterative procedure to obtain the design. Not only is the use of the toolkit simple but it is possible to print out a randomized plan for an experiment. The following is an example of a randomized plan for a resolvable IBED with $v = 36$, $k = 6$, and $r = 4$ constructed by the toolkit *GENDEX* where the rows are the incomplete blocks:

```
11  4 27  6  2  8      13 16  3 12 19 11
29  3 33 30 35 34      17 29 25 33 27 32
17 24 13 16 23  7       9 20 30 15 28 22
10 26  5 19 25 22       6 34  7 35 26 23
14  9 20 21 31 36      31  2 14  1 10 21
 1 18 32 12 28 15       8 24  5  4 36 18

 8 20 23  3 22  4      20  9 33 26 14 24
29 15  1 36 14 27      21 29 18 15 36 13
25 10  9 13 33  7      28 35 19  6 32 23
32 30 34 16 28 21       1 22 34  5  4  7
12 18  5 35  6  2       8 30 17  2 16 10
26 31 19 11 24 17       3 27 25 11 31 12
```

The intrablock efficiency of this design is 99.92% of the optimal value possible. Let R_i be the ratio of the number of replicates where an effect E_i is unconfounded with incomplete blocks to the total number of replicates r. The sum of R_iE_i for all effects divided by the number of effects is denoted as the intrablock efficiency factor. The efficiency is relative to a randomized complete block design with the *same* residual (error) mean square as the IBED. A toolkit is especially useful and labor-saving for many large plant breeding trials that can have up to thousands of genotypes in one trial. Some software packages useful for constructing a plan and a randomized plan are *GENDEX* [16–19], *Alpha+*, **SAS**, and *ECHIP*. Each of the toolkits has its own limitations for values of v and r.

The class of augmented experiment designs (AEDs), was developed for screening experiments involving large numbers of new and untried treatments [3, 8, 9, 12]. AEDs are useful for screening genotypes, herbicides, pesticides, drugs, etc. The treatments in an AED are divided into two groups, the first being the checks or standards and the second being the new or augmented treatments. The checks are considered as fixed effects, while the new are generally considered as random effects. To form an AED, an ED is selected for the c checks. Then, the blocks or the rows and/or columns are expanded to accommodate the n new treatments. The new treatments are usually included only once in an AED owing to a limited amount of material or because of a large value of n. The AED is always an IBED with regard to the new treatments.

The Latin square design has v treatments arranged in v columns and v rows in such a manner that each treatment appears once in each of the rows and once in each of the columns. This has been discussed in the literature at least since the time of the famous mathematician Euler. For use in experiments, the rows and columns refer to two sources of variation and not necessarily to a row-column lattice. The two sources of variation could be complete blocks and order within the blocks, stores and time periods, etc. Generalization of the Latin square has led to many other designs. One such is to have v columns and fewer or more than v rows. The so-called Youden design is a p-row by c-column design for $v = c$, and the treatments and the c columns form a balanced block design. Another is the 'F-square' which allows one or more treatments to occur more than once in each row and column. If $vr = cp$ (r is the number

of times each treatment is replicated), then a general row–column design can be constructed such that the restriction on the number of times a treatment occurs in a row or column is lifted. The row-column module of a toolkit may be used to obtain randomized plans for row-column designs.

In a response to Latin square type designs for large v and $r < v$, Yates [24] developed the lattice square design for $v = k^2$ treatments in k rows and k columns within each of r complete blocks. This design controls heterogeneity in the same manner as the Latin square but without excessive replication. Design construction and randomization for $v = ks = pc$ may be accomplished using the resolvable row–column module of a toolkit. The following is an example of an optimal resolvable row–column design, lattice rectangle design, for $v = 42$, $p = 6$, $c = 7$, and $r = 2$ constructed from a toolkit:

Replicate 1							Replicate 2						
7	42	19	6	25	3	18	37	39	11	22	28	12	18
31	28	20	33	41	4	16	24	13	1	41	6	31	38
15	38	11	8	27	14	29	8	16	25	3	35	40	34
34	21	13	12	9	17	2	20	42	17	29	23	27	30
23	40	32	5	37	39	1	7	26	33	21	14	10	32
22	36	35	30	26	24	10	2	15	36	5	9	19	4

When v exceeds the capacity of a toolkit, it will be necessary to construct a row–column or resolvable row–column plan and then to perform a randomization in order to obtain the plan for an experiment. A simple procedure for obtaining resolvable row–column designs is described by Federer [7]. Two examples are used to illustrate the procedure, the first for $v = 30$, $c = 5$, $p = 6$, and $r = 6$, and the second for $v = 228$, $c = 12$, $p = 19$, and $r = 5$. The latter case was for a plant breeding variety trial for which the incomplete block size could have been 2, 3, 4, 6, 12, 19, 57, or 76. The number of rows and columns could have been 12 and 19, 6 and 38, 4 and 57, 3 and 76, or 2 and 114. The experimenter actually selected an IBED with $k = 4$.

An appropriate model which accounts for the experimental variation present in an experiment should be selected for the statistical analysis of the data from an experiment. Standard textbook analyses may be inappropriate, inadequate, and/or misleading in accounting for the variability present in an experiment. Exploratory model selection is feasible owing the current availability of computer software. Using available software packages, it is possible to obtain quickly and easily the computations for several models. A comparison of the models may then be made to determine which one best accounts for the variability present. Bozivich et al. [2], Box and Cox [1], Federer [8], and Federer et al. [11, 13] have given guidance for model selection.

In designing an experimental plan and summarizing the resulting data, the following axioms should be followed [4]:

Axiom 1 Design for the particular experiment under consideration; do not experiment for the design (i.e. change the experiment to fit a known design).

Axiom 2 Use the minimum blocking possible to control the heterogeneity present in an experiment.

Axiom 3 Select an appropriate model that accounts for the variation present in the data from the experiment.

References

[1] Box, G.E.P. & Cox, D.R. (1964). An analysis of transformations (with discussion), *Journal of the Royal Statistical Society, Series B* **26**, 211–252.

[2] Bozivich, H., Bancroft, T.A. & Hartley H.O. (1956). Power of analysis of variance test procedures for incompletely specified models, *Annals of Mathematical Statistics* **27**, 1017–1043.

[3] Federer, W.T. (1961). Augmented designs with one-way elimination of heterogeneity, *Biometrics* **17**, 447–473.

[4] Federer, W.T. (1984). Principles of statistical design with special reference to experiment and treatment design, in *Statistics, An Appraisal*, H.A. David & H.T. David, eds, Iowa State University Press, Ames, pp. 77–104.

[5] Federer, W.T. (1993). *Statistical Design and Analysis for Intercropping Experiments, Two Crops*, Vol. I, Springer-Verlag, New York.

[6] Federer, W.T. (1995). A simple procedure for constructing experiment designs with incomplete block sizes of 2 and 3, *Biometrical Journal* **37**, 899–907.

[7] Federer, W.T. (1998). A simple procedure for constructing resolvable row-column designs. BU-1438-M in the Technical Report Series of the Department of Biometrics, Cornell University, Ithaca.

[8] Federer, W.T. (1998). Recovery of interblock, intergradient, and intervariety information in incomplete block and lattice rectangle designed experiments, *Biometrics* **54**, 471–481.

[9] Federer, W.T. & Raghavarao, D. (1975). On augmented designs, *Biometrics* **31**, 29–35.

[10] Federer, W.T. & Speed, T.P. (1987). Measures of block design efficiency recovering interblock information, Trans. Fourth Army Conference on Applied Mathematics and Computing, ARO-Report 87-1, US Army Research Office, Durham, pp. 1–29.

[11] Federer, W.T., Crossa, J. & Franco, J. (1998). New forms of spatial analysis with mixed model effects and exploratory model selection. BU-1406-M in the Technical Report Series of the Department of Biometrics, Cornell University, Ithaca.

[12] Federer, W.T., Nair, R.C. & Raghavarao, D. (1975). Some augmented row-column designs, *Biometrics* **31**, 361–373.

[13] Federer, W.T., Newton, E.A. & Altman, N.S. (1997). Combining standard block analyses with spatial analyses under a random effects model, in *Modelling Longitudinal and Spatially Correlated Data – Methods, Applications, and Future Directions*, T.G. Gregoire et al., eds, Springer-Verlag, New York, pp. 373–386.

[14] Federer, W.T., Nshinyabakobeje, S. & Nguyen, N-K. (1998). GENDEX for constructing experiment designs. BU-1433-M in the Technical Report Series of the Department of Biometrics, Cornell University, Ithaca.

[15] Khare, M. & Federer, W.T. (1981). A simple construction procedure for resolvable incomplete block designs for any number of treatments, *Biometrical Journal* **23**, 121–132.

[16] Nguyen, N-K. (1993). A toolkit for generating designs of experiments. `http://designcomputing.hypermart.net/gendex`

[17] Nguyen, N-K. (1993). An algorithm for constructing optimal resolvable incomplete block designs, *Communications in Statistics – Simulation and Computation* **22**, 911–923.

[18] Nguyen, N-K. (1994). Construction of optimal block design by computer, *Technometrics* **36**, 300–307.

[19] Nguyen, N-K. (1997). Construction of optimal row–column designs by computer, *Computing Science and Statistics* **28**, 471–475.

[20] Nguyen, N-K. & Williams, E.R. (1993). An algorithm for constructing optimal resolvable row–column designs, *Australian Journal of Statistics* **35**, 363–370.

[21] Patterson, H.D. & Williams, E.R. (1976). A new class of resolvable incomplete block designs, *Biometrika* **63**, 83–92.

[22] Patterson, H.D., Williams, E.R. & Patterson, L. (1985). A note on resolvable incomplete block designs, *Biometrical Journal* **27**, 75–79.

[23] Shafiq, M. & Federer, W.T. (1979). Generalized N-ary block designs, *Biometrika* **66**, 115–123.

[24] Yates, F. (1936). A new method of arranging variety trials involving a large number of varieties, *Journal of Agricultural Science* **26**, 424–455.

[25] Yates, F. (1940). Lattice squares, *Journal of Agricultural Science* **30**, 672–687.

(*See also* **Nested experimental designs**)

WALTER T. FEDERER & NAM-KY NGUYEN